Lecture 8 : Almost sure limits for sums of independent random v

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This set of notes is a revision of the work of Animesh Kumar, 2002.

8.1 Showing almost sure convergence via subsequences

General settings/notation: let $S_n = X_1 + ... + X_n$. The X_i's are assumed independent, all defined on some probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. Sometimes, we assume the X_i are identically distributed.

An important technique for showing almost sure convergence from convergence in probability is to consider subsequences. We first note a few general facts about the various types of convergence we know:

1. If $Y_n \to Y$ a.s. then $Y_n \to Y$ in $\mathbb{P}$.
2. If $Y_n \to Y$ in $\mathbb{P}$ then there exists a fixed increasing subsequence n_k such that $Y_{n_k} \to Y$ a.s.
3. $Y_n \to Y$ in $\mathbb{P}$ iff for every subsequence n_k there exists a further subsequebce n'_k so that $Y_{n'_k} \to Y$ a.s.

Proofs of 2 and 3 are in the textbook. We first begin with a technique which uses the information about almost sure convergence of a subsequence of a sequence of random variables, and then somehow getting control over a maximum. Let us now describe the technique.

One can prove $Y_n \to Y$ a.s. by first showing $Y_{n_k} \to Y$ a.s. for some n_k (we choose n_k) and then getting control over

$$M_k = \max_{n_k \le m < n_{k+1}} |X_m - X_{n_k}|$$

In particular we must be able to show that $M_k \to 0$ a.s. because if $\omega \in \Omega$ is such that both $X_{n_k}(\omega) \to 0$ and $M_k(\omega) \to 0$ then we get (using the triangle inequality and th fact that the max is greater than the elements of set over which maximum is taken)

$$X_m(\omega) \to X(\omega),$$

so if $M_k \xrightarrow{a.s.} 0$, then the above holds almost surely. To illustrate how to use the technique, we start with the example of SLLN with a second moment condition.

Theorem 8.1 *If $X, X_1, X_2, ...$ are i.i.d. random variables with $E(X) = \mu$, $E(X^2) < \infty$, and $S_n := X_1 + X_2 + ... + X_n$, then*

$$\frac{S_n}{n} \to E(X) \ a.s. \tag{8.1}$$

Proof: First we find a subsequence converging almost surely to the mean. For that we use two tools:

- convergence in probability; and
- the Borel-Cantelli lemma.

Without loss of generality, we can assume that $E(X) = 0$. From Chebyshev's inequality we get

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \epsilon\right) < \frac{E(X^2)}{n\epsilon^2}.$$

This means that $\frac{S_n}{n} \to 0$ in $\mathbb{P}$. Notice that $\sum_k \frac{1}{k^2}$ converges to a finite value, therefore for the subsequence $n_k = k^2$ we get, using the Borel-Cantelli lemma,

$$\mathbb{P}\left(\left|\frac{S_{n^2}}{n^2}\right| > \epsilon \text{ i.o.}\right) = 0,$$

which means that $\frac{S_{n^2}}{n^2} \to 0$ a.s.

Now let us try to control M_k as defined above. For convenience we define

$$D_n := \max_{n^2 \le k < (n+1)^2} |S_k - S_{n^2}|$$

for $n^2 \le k < (n+1)^2$. We have $|S_k| \le |S_{n^2}| + D_n$ and $\frac{1}{k} \le \frac{1}{n^2}$, so we have the following inequality:

$$\left|\frac{S_k}{k}\right| \le \left|\frac{S_{n^2}}{n^2}\right| + \frac{D_n}{n^2}.$$

Finally, using the definition of M_k, we get the following:

$$\begin{aligned} M_k &\le \max_{n^2 \le k < (n+1)^2} \left|\frac{S_k}{k}\right| + \left|\frac{S_{n^2}}{n^2}\right| \\ &\le 2\left|\frac{S_{n^2}}{n^2}\right| + \frac{D_n}{n^2}. \end{aligned}$$

So all we need to prove is that $\frac{D_n}{n^2} \to 0$ a.s. Let us define a new quantity $T_m = S_{n^2+m} - S_{n^2}$. Therefore,

$$\begin{aligned} D_n^2 &= \max_{1 \le m \le 2n} T_m^2 \\ &\le \sum_{m=1}^{2n} T_m^2. \end{aligned}$$

Taking expectations on both sides, we get that

$$\begin{aligned} E(D_n^2) &\le \sum_{m=1}^{2n} m\sigma^2 = n(2n+1)\sigma^2 \\ &\le 4n^2\sigma^2, \end{aligned}$$

where $E(X^2) = \sigma^2$. Hence we get that

$$\begin{aligned} \mathbb{P}\left(\left|\frac{D_n}{n^2}\right| > \epsilon\right) &\le \frac{E\left(\left(\frac{D_n}{n^2}\right)^2\right)}{\epsilon^2} \\ &\le \frac{4\sigma^2}{n^2\epsilon^2}. \end{aligned}$$

Applying the Borel-Cantelli lemma with the fact

$$\sum_n \mathbb{P}\left(\left|\frac{D_n}{n^2}\right| > \epsilon\right) < \infty$$

we get that $\frac{D_n}{n^2} \to 0$ a.s., which completes the proof. ∎

8.2 Kolmogorov's Maximal Inequality

Now we proceed to Kolmogorov's inequality. We formally state it as follows.

Theorem 8.2 (Kolmogorov's Inequality) *Let $X_1, X_2, ...$ be independent with $E(X_i) = 0$ and $\sigma_i^2 = E(X_i^2) < \infty$, and define $S_k = X_1 + X_2 + ... + X_k$. Then*

$$\mathbb{P}\left(\max_{1 \le k \le n} |S_k| \ge \epsilon\right) \le \frac{E(S_n^2)}{\epsilon^2}. \tag{8.2}$$

Proof: Decompose the event according to when we escape from the $\pm\epsilon$ strip. Let

$$A_k = \{|S_m| < \epsilon \text{ for } 1 \le m < k;\ |S_k| \ge \epsilon\}$$

In words, A_k is the event that the first escape out of the ϵ strip occurs at the kth step. Also notice that all these events are disjoint, and that $\bigcup_{k=1}^n A_k \{\max_{1\le k\le n} |S_k| \ge \epsilon\}$. Then,

$$E(S_n^2) \ge E\left(S_n^2 1\left(\bigcup_{k=1}^n A_k\right)\right) \sum_{k=1}^n E\left(S_n^2 1_{A_k}\right).$$

We can split $S_n^2 = S_k^2 + (S_n - S_k)^2 + 2S_k(S_n - S_k)$, and write

$$\begin{aligned} E\left(S_n^2 1_{A_k}\right) &= E\left(S_k^2 1_{A_k}\right) + E\left((S_n - S_k)^2 1_{A_k}\right) + E\left(2(S_n - S_k)S_k 1_{A_k}\right) \\ &\ge \epsilon^2 \mathbb{P}(A_k), \end{aligned}$$

where the first term is larger than ϵ^2, the second term is always positive, and the third term is the expectation of a product of two independent random variables witn mean 0.

Finally, we put this into the summation to get

$$E(S_n^2) \ge \sum_{k=1}^n \mathbb{P}(A_k)\epsilon^2 = \mathbb{P}\left(\max_{1\le k\le n} |S_k| \ge \epsilon\right)\epsilon^2,$$

which easily leads to the result. This completes the proof. ■

We observe that the inequality is valid for any sequence of r.v.'s $(X_1, ...X_n)$ such that

$$E\left(2(S_n - S_k)S_k 1_{A_k}\right) = 0.$$

This will lead to the definition in future lectures of a *martingale difference sequence.*